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Properties of Hida Processes on $\mathbb{R}^2$. 1. N-Hida Processes

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If E is an ordered set, we study the processes Y_t, $t \in E$, for which the vectorial spaces $\mathscr{E}_t$ generated by all the conditional expectations $E(Y_s | \mathscr{F}_t)$ for $s \geqslant t$ have finite dimensions $d(t) \leqslant N$. ($\mathscr{F}_t$ is some convenient filtration.) We first develop a geometrical approach in the general situation and give a "Goursat's representation" $Y_t = \sum f_i(t) M_i(t)$, where the $M_i(t)$ are martingales. We then restrict us to the cases $E = \mathbb{R}$ or $E = \mathbb{R}^2$ and give representations of the processes by the mean of stochastic integrals of "Goursat's kernels." The special case when Y_t is the solution of a differential equation is considered.

INTRODUCTION

In this article we study square integrable centered processes Y_t where t moves along a filtering ordered set E (in fact $\mathbb{R}$ or $\mathbb{R}^2$) and verifies the following property: Given an increasing family of fields $\{\mathscr{F}_t, t \in E\}$, i.e., a filtration and Y_t being $\mathscr{F}_t$-adapted, for every couple of elements (s, s') belonging to E such that $s \leqslant s'$, the closed vectorial space (for the L^2norm) generated by the expectations of Y_t conditional to $\mathscr{F}_s$ when $t > s'$, written as

$$\mathscr{E}_s = \operatorname{sp}\{E(Y_t | \mathscr{F}_s), t \geqslant s'\},$$

is independent of s' and has a bounded dimension d_s.

Hida [5] introduced this notion under the name of "Markov N-ple" when d_s is a constant equal to N and for gaussian processes with an index in $\mathbb{R}$ of multiplicity equal to 1 expressed by the representation

$$Y = \int^t F(t, u)\, B(du) \tag{1}$$

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where $F(t, \cdot)$ is deterministic in $L^2(]-\infty, t], du)$ and B is a normalised brownian measure.

Hereafter, we shall refer to this property as the "*Hida-bounded*" *property*, or in the particular case where d_s is a constant equal to N as the "N-Hida" property relative to the $\mathscr{F}_t$-filtration.

In this case Hida proves that if a Y_t process allowing representation (1) is N-Hida, then $F(t', u)$ is a Goursat kernel of order N; i.e.,

$$F(t, u) = \sum_{\alpha=1,N} f_\alpha(t)\, g_\alpha(u) \tag{2}$$

(with some conditions of regularity on f_α, g_α).

Pitt [7] generalised this property to the case of gaussian processes with an index in $\mathbb{R}$, without any conditions on the representation or the multiplicity of this process, showing that if Y_t is N-Hida it allows a Goursat N-representation

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t)\, M_\alpha(t) \tag{3}$$

where $\{f_\alpha\}$ satisfies some regularity conditions, and $\{M_\alpha\}$ is a regular martingale. (It is then proved that the multiplicity of the process is bounded by N.)

The latter representation clearly illustrates the notion of the "N-Hida property" of a process, an intermediary property situated between a Markov property and a martingale-type property. The property is similar to a Markov property in various ways, without being generally a Markov property: if $N = 1$ we have a simple Markov property between the future and the past for every time t; for any N, if $t_1 < t_2 < \cdots < t_{n+1}$ are $(N+1)$ separate points, then the $N+1$ conditional expectations $\{E(Y_{t_j} \mid \mathscr{F}_t), j = 1, N+1\}$ are dependent for every t smaller than t_j; the information to predict $Y_{t_{N+1}}$ from $\mathscr{F}_t$ being summed up by the N variables $E(Y_{t_j} \mid \mathscr{F}_t), j = 1, N)$.

Let us note that $t_1, \ldots, t_N$ may be chosen arbitrarily close to t, and therefore in a certain way $\mathscr{E}_s$ is generated by the germ of the process in t: in particular, under adequate appropriate assumptions on the derivability of Y_t, $\mathscr{E}_s$ will be generated by the process and its derivatives at point s ([6, Sect. 3] in the case of $\mathbb{R}$).

The property is also close to a martingale property since an "N-Hida" property may be expressed by an N-representation from martingales.

In the following pages we shall extend and develop in several directions the properties of the processes proving the "Hida" property.

(a) Firstly the E-index set need no more necessarily be $\mathbb{R}$ but a filtering ordered set, typically $\mathbb{R}$ (total order) or $\mathbb{R}^2$ (partial order): the new difficulties arising mainly from the fact that the order is partial (Sects. 1, 2).

(b) Secondly, we shall not assume the process to be gaussian, but only centered and in L^2.

(c) Also, the family $\{\mathscr{F}_t, t \in E\}$ is general: it is not necessarily a natural filtration of the Y_t process (in (4) and (7), $\mathscr{F}_t$ is the natural field of the process). $\{\mathscr{F}_t\}$ is merely assumed to be increasing and Y_t $\mathscr{F}_t$-adapted. This more general situation does not create extra difficulties but provides new results in several cases. Therefore, in the same way that we usually talk of a martingale with reference to a filtration, we shall talk of the Hida property with reference to a given filtration, thus bringing this property closer to that of martingales, and hence moving further away from markovian properties, where it is natural to work with the natural filtration.

When $\mathscr{F}_t = \mathscr{F}_t(Y)$ we shall simply say Hida property. Let us note that if Y_t is Hida-bounded in relation to $\mathscr{F}_t$, it is also Hida (the $\mathscr{E}_s$ spaces are consequently of smaller dimension) relative to $\mathscr{F}_t(Y)$.

(d) We shall not assume that the dimension of $\mathscr{E}_s$, i.e., the order of the Hida property in s, is constant, but we shall simply assume that it is bounded and for this reason we shall talk of Hida-bounded property: the dimension of $\mathscr{E}_s$ increasing in s, a "progressive" representation of the process will be obtained from martingales taking "birth" along certain decreasing lines (points in the case of $\mathbb{R}$, Sect. 2).

(e) We shall particularly develop the study of the Hida properties for processes with index in $\mathbb{R}^2$, and be led to introduce the notion of Hida-bounded in the sense 1: "$(Y_{st})_{s \in \mathbb{R}}$ is Hida-bounded to a parameter relative to filtration (nonnatural) $\mathscr{F}^1_s = \bigvee_{u \leqslant s} \mathscr{F}_{uv}$". This notion will be associated to the notion of 1-martingale relative to filtration $(\mathscr{F}^1_s)$.

Under certain conditions we show that the Hida properties in both directions imply the Hida property relative to $(\mathscr{F}_{st})$.

In Section 2, we shall study this type of property for much larger pasts defined from the σ-field (in the natural case) generated by $\{Y_{Z'}, Z' \not\succ Z\}$ and finally in the case of certain problems of statistical mechanics where the past is outside a rectangle.

1. Study of the N-Hida Property for a Filtering Set

We shall first extend a certain number of classical definitions to the case of a filtering ordered set.

Let E be an ordered set ($\leqslant$ will denote its order) filtering to the left and to the right (i.e., $\forall s, t\ \exists r, v$ such that $r \leqslant s \leqslant v$, $r \leqslant t \leqslant v$).

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $(\mathscr{F}_t)_{t \in E}$ a filtration of $(\Omega, \mathscr{F})$, that is to say an increasing sequence of subfields of $\mathscr{F}$; each $\mathscr{F}_t$ will be assumed complete. Throughout this paper, $\{Y_t, t \in E\}$ will be a process of

$L^2(P)$, adapted in the sense that Y_t is $\mathscr{F}_t$-measurable (i.e., $\mathscr{F}_t(Y) \subset \mathscr{F}_t$ where $\mathscr{F}_t(Y)$ is the field generated by $\{Y_s, s < t\}$.

For every couple $s \leqslant t$, let us denote

$$\mathscr{E}_s^t = \mathrm{sp}\{E(Y_v \mid \mathscr{F}_s), t \leqslant v\}$$

(where sp$(\cdot)$ indicates the closing of the vectorial space in L^2 generated by $\cdot$).

Let us note that if $s \leqslant t \leqslant t'$

$$\mathscr{E}_s^{t'} \subset \mathscr{E}_s^t.$$

The following definition then adopts the expression "Markov–N–ple" given by Hida (cf. [5, 8]).

DEFINITION 1.1. The process Y_t will be called N-Hida relative to $(\mathscr{F}_t)$ if for every couple $s < t$, $\mathscr{E}_s^t$ has an N dimension. The process will be said to be N-Hida if it is N-Hida relative to its natural field.

From the preceding remark one infers

$$\text{if} \quad s \leqslant t \leqslant t' \qquad \text{then} \quad \mathscr{E}_s^{t'} = \mathscr{E}_s^t = \mathscr{E}_s^s.$$

In the following pages we shall write $\mathscr{E}_s = \mathscr{E}_s^s$.

DEFINITION 1.2. A system of N real, deterministic functions $f_\alpha(t)$, $\alpha = 1$, N, will be said to form a Tchebychev system on a subspace A of E if there exist N points of A, t_j, $j = 1, N$, such that the matrix $(f_\alpha(t_j))_{\alpha=1,N;j=1,N}$ is regular.

DEFINITION 1.3. $M(t) = {}^t(M_1(t),..., M_n(t))$ will be called a vectorial martingale for the $\mathscr{F}_t$ filtration if

$$\forall s \leqslant t \qquad \forall \alpha = 1, N \qquad E(M_\alpha(t) \mid \mathscr{F}_s) = M_\alpha(s).$$

$M(t($ will be said regular in L^2 if for every t the random variables $M_\alpha(t)$, $\alpha = 1, N$, are linearly independent in L^2.

For a martingale of this type we shall note $\mathscr{F}_t(M)$ the field generated by

$$\{M_\alpha(s), s \leqslant t; \alpha = 1, N\}.$$

DEFINITION 1.4. The Y_t process will be said to allow a Goursat representation of order N relative to $\mathscr{F}_t$ if Y_t may be written in the form

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t)\, M_\alpha(t) \tag{1.1}$$

where

(a) the functions $f_\alpha(t)$, $\alpha = 1, N$ form a Tchebychev system on every subspace $A_s = \{t, s \leqslant t\}$,

(b) $M(t) = {}^t(M_1(t),..., M_n(t))$ is a regular vectorial martingale for the $\mathscr{F}_t$ filtration.

$\mathscr{F}_t(Y) \subset \mathscr{F}_t(M) \subset \mathscr{F}_t$ is always true. The (1.1) representation will be called proper if

$$\forall t \qquad \mathscr{F}_t(Y) = \mathscr{F}_t.$$

THEOREM 1.1. *Let Y_t, $t \in E$ be an $\mathscr{F}_t$-adapted process in L^2. Y_t is N-Hida relative to $\mathscr{F}_t$ if and only if it allows an N order Goursat representation relative to $\mathscr{F}_t$.*

Proof. Let us first prove the sufficient condition. If we have (1.1) since the M_α are martingales

$$E(Y_t \mid \mathscr{F}_s) = \sum_{\alpha = 1,N} f_\alpha(t)\, M_\alpha(s).$$

$\mathscr{E}_s^t$ is therefore generated by $M_\alpha(s)$, $\alpha = 1, N$ which is a free family since $M(s)$ is regular. Since by assumption $t_j > s$, $j = 1, N$ can be chosen so that $(f_\alpha(t_j))_{\alpha = 1,N; j = 1,N}$ is regular, $\mathscr{E}_s^t$ is of dimension N.

Let us prove the necessary condition.

Let $s \leqslant t$ and let us call T_{st} the restriction to $\mathscr{E}_t$ of the linear operator $E(\cdot \mid \mathscr{F}_s)$ defined on L^2. Since Y_t is N-Hida, $v_j > t$, $j = 1, N$ exist such that $\mathscr{E}_s = \{E(Y_{v_j} \mid \mathscr{F}_s),\ j = 1, N\}$ forms a base of $\mathscr{E}_s$. We have $E(Y_{v_j} \mid \mathscr{F}_s) = T_{st}(Y_{v_j} \mid \mathscr{F}_t)$ and therefore $\mathscr{E}_t = \{E(Y_{v_j} \mid \mathscr{F}_t)\}$ forms a base of $\mathscr{E}_t$ since $\dim \mathscr{E}_t = \dim \mathscr{E}_s = N$. T_{st} being bijective, let us call T_{ts} its inverse.

Now let s and t be nonordered, v a majorant of s and t. We write $T_{st} = T_{sv} \circ T_{vt}$. Let us show that this definition is independent of v. Let $v_j, j = 1, N$ be N points such that $v_j > v$ and that $\mathscr{E}_v = \{E(Y_{v_j} \mid \mathscr{F}_v), j = 1, N\}$ is a base of $\mathscr{E}_v$. If $\mathscr{E}_u = \{E(Y_{v_j} \mid \mathscr{F}_u),\ j = 1, N\}$, then according to what has been stated above, $\mathscr{E}_s$ and $\mathscr{E}_t$ are respective bases of $\mathscr{E}_s$ and $\mathscr{E}_t$. Moreover

$$T_{vt} E(Y_{v_n} \mid \mathscr{F}_t) = E(Y_{v_n} \mid \mathscr{F}_v)$$
$$T_{sv} E(Y_{v_n} \mid \mathscr{F}_v) = E(Y_{n_n} \mid \mathscr{F}_s);$$

therefore

$$T_{sv} \circ T_{vt} E(Y_{v_n} \mid \mathscr{F}_t) = E(Y_{v_n} \mid \mathscr{F}_s)$$

and hence $T_{sv} \circ T_{vt}$ is an isomorphism of $\mathscr{E}_t$ onto $\mathscr{E}_s$ independent of v.

If (r, s, t) is a triplet of E, then we have

$$T_{ts} \circ T_{sr} = T_{tr}$$

Actually, let v be a majorant of (r, s, t), and v_j, $j = 1, N$ a sequence such that $\mathscr{E}_v$ is a base of $\mathscr{E}_v$. Then $\mathscr{E}_r$, $\mathscr{E}_s$, and $\mathscr{E}_t$ are free in $\mathscr{E}_r$, $\mathscr{E}_s$, and $\mathscr{E}_t$ and, as formerly, $T_{sr} = T_{sv} \circ T_{vr}$ transforms the $\mathscr{E}_r$ base into the $\mathscr{E}_s$ base and $T_{ts} = T_{tv} \circ T_{vs}$ transforms $\mathscr{E}_s$ into $\mathscr{E}_t$. Therefore $T_{tr} = T_{ts} \circ T_{sr}$. Let $t_0 \in E$ be fixed, and $M_1(t_0),..., M_n(t_0)$ be a base of $\mathscr{E}_{t_0}$. Let us write $M_\alpha(t) = T_{tt_0} M_\alpha(t_0)$, $\alpha = 1, N$.

Since $T_{t_0 t}$ is a bijection, $\{M_\alpha(t)\}$ is free and hence the covariance matrix of $M(t)$ is regular. Furthermore, if $s < t$,

$$\begin{aligned} E(M_\alpha(t) \mid \mathscr{F}_s) &= T_{st} M_\alpha(t) = T_{st} \circ T_{tt_0} M_\alpha(t_0) \\ &= T_{st_0} M_\alpha(t_0) = M_\alpha(s); \end{aligned}$$

therefore M_α is a martingale.

Since $Y_t \in \mathscr{E}_t$, we may write $Y_t = \sum_{\alpha=1,N} f_\alpha(t) \, M_\alpha(t)$. Let $v > t$, $v_j > v$, $j = 1, N$, with $\{E(Y_{v_j} \mid \mathscr{F}_t), j = 1, N\}$ being free. Since

$$E(Y_{v_j} \mid \mathscr{F}_t) = \sum_{\alpha=1}^{N} f_\alpha(v_j) \, M_\alpha(t)$$

the matrix $(f_\alpha(v_j))_{\alpha=1,N; j=1,N}$ is regular.

Remarks. (1) The assumption used is not "E filtering on the left" but "for every triplet (r, s, t) of points in E, a majorant v can be found, as well as v_j, $j = 1, N$ such that $\{E(Y_{v_j} \mid \mathscr{F}_u), j = 1, N)$ is free, for $u \in \{r, s, t\}$".

(2) If $\{\tilde{M}_1(t_0),..., \tilde{M}_n(t_0)\}$ is another base of $\mathscr{E}_{t_0}$, the change of base may be written

$$\tilde{M}_\alpha(t_0) = \mathscr{C} M_\alpha(t_0)$$

and on every space $\mathscr{E}_t$ we have $\tilde{M}_\alpha(t) = T_{tt_0} \circ \mathscr{C} \circ T_{t_0 t} M_\alpha(t)$. And if

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t) \, M_\alpha(t) = \sum_{\alpha=1,N} \tilde{f}_\alpha(t) \, \tilde{M}_\alpha(t)$$

we have

$$f(t) = T_{tt_0} \circ \mathscr{C} \circ T_{t_0 t} \tilde{f}(t).$$

Therefore except for base change one single representation exists (independent of t). Two representations are therefore linear-equivalent.

2. Study of the N-Bounded Hida Property

Using the symbols of the former paragraph, we give

Definition 2.1. We shall say that a process Y_z satisfies the Hida property relative to $\mathscr{F}_z$ if $\forall z$, $\forall z' > z$, $\mathscr{E}_z = \mathscr{E}_z^{z'}$. We shall say that Y_z is Hida-bounded by N if, moreover,

$$\forall z \qquad \dim \mathscr{E}_z \leqslant N.$$

We study the case $\dim \mathscr{E}_z \leqslant N$. Since $d_z = \dim \mathscr{E}_z$ is an increasing function of z, areas $S_k = \{z, d_z = k\}$ may be defined, which form a partition of E such that if $z \in S_k$ for every $z' < z$, $z' \in S_h$, with $h \leqslant k$. In the following pages we shall build a Goursat representation, progressive in relation to the order, by defining birth surfaces of martingales which are boundaries for S_k. The method can be applied for every space E, a filtration, and on some points of detail, with a topology compatible with the order. Yet, to make things more concrete, most of our work will be carried out on $\mathbb{R}^2$ presenting a partial order defined by $z = (s, t) \leqslant z' = (s', t')$ iff $s \leqslant s'$ and $t \leqslant t'$.

The same order is used for the martingale theory on $\mathbb{R}^2$ [1, 11, 4]. We shall write $z \ll z'$ for $s < s'$ and $t < t'$ and $z \curlywedge z'$ for $s < s'$ and $t' < t$. The $\mathscr{F}_z$ filtrations taken into consideration are increasing for the order. A line of $\mathbb{R}^2$ will be said to be decreasing if no pair of points (z, z') exist on the graph with $z < z'$. We shall write $z < \gamma$ if $z' \in \gamma$ exists with $z < z'$, and $\gamma < z$ if $z' \in \gamma$ exists with $z' < z$.

We shall now prove the following theorem:

Theorem 2.1. *If Y_z is Hida-bounded by N, if $S_k = \{z, d_z = k\}$, and if $\gamma_k = \bar{S}_k \cap \bar{S}_{k+1}$, then an $M(z)$ vectorial martingale on $\mathbb{R}^2$, $M(z) = (M_\alpha(z))_{\alpha = 1-N}$, and a function $I : z \to \mathscr{P}(\{1, \ldots, N\})$ exist such that*

- $z < z' \Rightarrow I(z) \subseteq I(z')$,
- *$I(z)$ is constant on every connected component of S_k,*
- card $I(z) = k$ *on* S_k,
- *$(M_\alpha(z))_{\alpha \in I(z)}$ is a base of $\mathscr{E}_z$,*

and in particular $Y(z) = \sum_{\alpha \in I(z)} f_\alpha(z) M_\alpha(z)$ where $(f_\alpha(z_j))_{\alpha = 1,k; j = 1,k}$ is regular when an appropriate choice of separate points in any connected component of S_k is made when S_k has a nonempty interior.

Theorem 2.2. *If $\mathbb{R}^2$ is substituted by $\mathbb{R}$ in the above assertion, γ_k is then reduced to one point at the most, which is the upper limit of the S_k interval* *when* $S_k \neq \varnothing$. *For* $t \in S_k$, *we have* $I(t) = \{1,..., k\}$ *and therefore* $Y(t) = \sum_{\alpha=1,k} f_\alpha(t) M_\alpha(t)$.

Proof. We shall only prove Theorem 2.1, Theorem 2.2 being very much simpler, since a nontotal order leads to cases where S_k is not connected.

LEMMA 2.1. *If* z *and* z' *belong to the same connected component* $\mathscr{C}$ *from the interior of an area* S *bounded by two decreasing lines* (*i.e.*, $S = \{z, \gamma < z < \gamma'\}$), *a monotonous line in steps formed by a finite number of segments joining* z *and* z' *exists.*

This result can easily be checked in a standard manner. A complete proof can be found in [10].

If $z \in \mathscr{C}$, $z' \in \mathscr{C}$, $z < z'$, we know according to paragraph 1 how to define the bijection $T_{zz'} : \mathscr{E}_{z'} \to \mathscr{E}_z$ and we may write

$$T_{z'z} = T_{zz'}^{-1}.$$

Using the lemma stated above let us define $T_{z'z}$ for every nonordered pair (z', z). If $c = (z, z_1, z_2,..., z_p, z')$ is a line in steps joining z to z' let us write

$$T_{zz'}(c) = T_{zz_1} \circ T_{z_1z_2} \circ \cdots \circ T_{z_{p-1}z_p} \circ T_{z_pz'}.$$

Let us show that this definition does not depend on the line in steps chosen. Let us at first assume that $z_1 = z \vee z'$ and $\tilde{z}_1 = z \wedge z'$ belong to $\mathscr{C}$, let $X' \in \mathscr{E}_{z'}$; there exists only one X_1 in $\mathscr{E}_{z_1}$ (which has the same dimension) such that

$$X' = T_{z'z_1} X_1.$$

The $T_{zz'}$ operator defined by using the line which passes by z_1 will therefore be such that

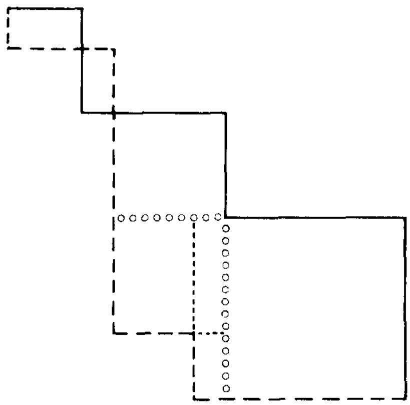
$$X = T_{zz'} X' = T_{zz_1} X_1.$$

We also have

$$\tilde{X}_1 = T_{\tilde{z}_1z'} X = T_{\tilde{z}_1z_1} X_1 = T_{\tilde{z}_1z} T_{zz_1} X_1 = T_{\tilde{z}_1z} X$$

and the $T_{zz'}$ operator defined by using the line which passes by $\tilde{z}_1$ will be such that $\tilde{T}_{zz'} X' = X$ and therefore $T_{zz'} = \tilde{T}_{zz'}$.

Now let us consider the general case of two points $z, z' \in \mathscr{C}$.



If z and z' are joined by any two step lines, one may go from one line to the other by repeating a finite number of times the opertation which consists of "turning a step upside down." Since none of these operations affect the definition of $T_{zz'}(c)$, the latter is independent of (c). The consequences are

LEMMA 2.2. *If one takes $z_0 \in \mathscr{C}$ and $(M_1(z_0),..., M_k(z_0))$ a base of $\mathscr{E}_{z_0}$, for every z of $\mathscr{C}$ one may write*

$$\forall \alpha = 1, k \qquad M_\alpha(z) = T_{zz_0} M_\alpha(z_0)$$

and $(M_1,..., M_k)(z)$ will be a regular vectorial martingale on $\mathscr{C}$.

We now want to construct a base $(M_1,..., M_n)$ of S_n so that for every connected component $\mathscr{C}$ of S_k a part $I_{\mathscr{C}}$ of $\{1,..., N\}$ exists such that $(M_\alpha, \alpha \in I)$ be a base of $\mathscr{E}_z$ if $z \in \mathscr{C}$.

Let us fix a point z_N in S_N. We shall write $\mathscr{E} = \mathscr{E}_{z_N}$ and for every z, we shall define $T_z : \mathscr{E} \to \mathscr{E}_z$. For that let ξ be an upper bound common to z_N and z. We set $T_z = T_{z\xi} \circ T_{\xi z_N}$ (when T_{zz_N} exists, $T_z = T_{zz_N}$) and it can easily be proved that this definition is independent of ξ. It can also be shown that if $T_{z'z}$ exists, $T_{z'} = T_{z'z} \circ T_z$.

LEMMA 2.3. *Let $(M_1(z),..., M_p(z))$ be a vectorial martingale on $\mathbb{R}^2$.*

(1) *If at one point z, $\{M_1(z),..., M_p(z)\}$ is a free system, then $z < z' \Rightarrow \{M_1(z'),..., M_p(z')\}$ is free.*

(2) *If at one point z, $\{M_1(z),..., M_p(z)\}$ is a generator of $\mathscr{E}_z$, then $z'' < z \Rightarrow \{M_1(z''),..., M_p(z'')\}$ generates $\mathscr{E}_{z''}$.*

The proof of this lemma is obvious, by using the fact that if $z < z'$, $T_{zz'}$ goes from $\mathscr{E}_{z'}$ onto $\mathscr{E}_z$.

Denoting always by $S_k = \{z; \dim \mathscr{E}_z = k\}$ and by $D_k = \{z; \dim \mathscr{E}_z \leqslant k\}$, we deduce from this lemma that, if D_k is not enpty and not all $\mathbb{R}^2$, D_k is the past of a decreasing line, say γ_k. We have $\gamma_{k-1} < z < \gamma_k \Rightarrow z \in S_k$, but each S_k may include more than one connected component. Let G be the union of all the γ_k and Θ the set of decreasing lines included in G.

We come now to the proof of Theorem 2.1.

(1°) Let us choose any point in S_m and let E_0 be any supplementary of Ker T_{z_m} in $\mathscr{E}$. T_{z_m} estblishes a bijection of E_0 on $\mathscr{E}_{z_m}$ and E_0 has therefore an m dimension. Let us choose for it any basis $(M_1,..., M_m)$ and define for all z:

$$M_\alpha(z) = T_z M_\alpha.$$

$(M_1(z),..., M_m(z))$ is a martingale, and one infers from Lemma 2.3 that it is regular.

(2°) We shall use a finite recurrence method: let us suppose that after some steps, we obtain a vectorial martingale $(M_1(z),..., M_k(z))$ on $\mathbb{R}^2$ such as, denoting for all z by $J(z)$ the set $\{\alpha; M_\alpha(z) \neq 0\}$, then $(M_\alpha(z))_{\alpha \in J(z)}$ is regular.

Let us denote by $\mathscr{R}_k = \{z; (M_1(z),..., M_k(z))$ does not generate $\mathscr{E}_z\}$. We have $\mathscr{R}_k \subseteq \mathscr{F}_{k-1}$ and $\mathscr{R}_k$ is not empty, except for $k = N$. Then there exists a line θ_k in Θ such that

$$z \in \mathscr{R}_k \Leftrightarrow \theta_k < z.$$

Let ξ_k be a point of one of the connected components, say $\mathscr{C}$, of $\mathscr{R}_{k+1} - G$, such that the boundary of $\mathscr{C}$ has a nonempty intersection with θ_k. Let F_k be

$$F_k = \mathrm{sp}(M_1,..., M_k) \subseteq \mathscr{E}$$

and E_k be any supplementary of F_k. T_{ξ_k} cannot be identically zero on E_k, otherwise, it would be a bijection between $\mathscr{E}$ and $\mathscr{E}_{\xi_k}$ and $(M_1(\xi_k),..., M_k(\xi_k))$ would generate $\mathscr{E}_{\xi_k}$. Let M_{k+1} be a nonzero vector in E_k and

$$\forall z \qquad M_{k+1}(z) = T_z M_{k+1}.$$

$(M_1(z),..., M_{k+1}(z))$ is a martingale in $\mathbb{R}^2$. If $z \notin \mathscr{R}_{k+1}$, $M_{k+1}(z) = 0$, but $M_{k+1}(\xi_k) \neq 0$. There exists a line ψ_{k+1} of Θ such that

$$z < \psi_{k+1} \Rightarrow M_{k+1}(z) = 0 \qquad \psi_{k+1} < z \Rightarrow M_{k+1}(z) \neq 0$$

M_{k+1} appears then when crossing ψ_{k+1}.

(3°) After $N - m$ steps, we obtain the vectorial martingale $(M_1(z),..., M_N(z))$. On S_N it is a basis of $\mathscr{E}_z$ and by Lemma 2.3

$$\forall z \qquad (M_1(z),..., M_N(z)) \text{ generates } \mathscr{E}_z.$$

Let us choose a basis of each $\mathscr{E}_z$ in the following manner:

— take $M_1(z)$

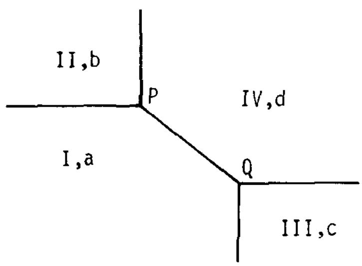
— take $M_\alpha(z)$ iff it forms a free system with the $M_\beta(z)$ already taken. Let us denote by $I(z)$ the set of indices chosen in z. If $z \in S_k$, card $I(z) = k$. From Lemma 2.2, we deduce that if z and z' belong to the same connected component of G^c, $I(z) = I(z')$, and from Lemma 2.3 that

$$z \leqslant z' \Rightarrow I(z) \subseteq I(z').$$

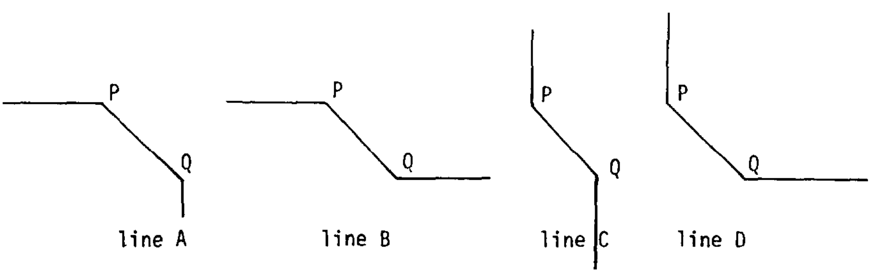
This achieves the proof of Theorem 2.1.

Note. As an example, the following situation is hence impossible: $z < z''$, $z' < z''$ and $I(z) = \{1, 2\}$, $I(z') = \{3, 4\}$, $I(z'') = \{1, 2, 3\}$. Actually, if $M_4(z'')$ is a linear combination of $M_1(z'')$, $M_2(z'')$ and $M_3(z'')$, this combination can be projected (by Lemma 2.3) on $\mathscr{E}_{z'}$, and gives that $M_4(z')$ is proportional to $M_3(z')$.

EXAMPLE. In the next scheme, we give the values of dim $\mathscr{E}_z$ in the four regions I, II, III, and IV. Suppose $a < b < d$ and $a < c < d$:



Let us consider the four lines



$\mathscr{R}_1$ is the future of the line A. Let $V_0 = (M_1(z),..., M_a(z))$ be a basis of $\mathscr{E}_z$ when $z \in I$. Choose as an example ξ_2 in II. To complete a basis of $\mathscr{E}_{\xi_2}$, we choose martingales $M_{a+1}(z),..., M_b(z)$. They then appear along A or B. If it is impossible to extract from $(M_1(z),..., M_b(z))$ a basis of $\mathscr{E}_z$ when $z \in$ III (in partiular if $b < c$), $\mathscr{R}_2$ shall be the future of line C. When completing the basis in a point $\xi_3 \in$ III, we introduce martingales which appear along C. If $\mathscr{R}_3 \neq \phi$, it is the future of D and we can complete the basis. (Note that we should have chosen directly a point ξ in IV.)

If V_{A}, V_{B}, V_{C}, and V_{D} are the sets of martingales which appear along A, B, C and D,

in I the basis is V_0

in II the basis is $V_0, V_{\mathrm{A}}, V_{\mathrm{B}}$

in III the basis is $V_0, V_{\mathrm{A}}, V_{\mathrm{C}}$

in IV the basis is $V_0, V_{\mathrm{A}}, V_{\mathrm{B}}, V_{\mathrm{C}}, V_{\mathrm{D}}$,

3. Processes on $\mathbb{R}$: Multiplicity

3.1. *Representation of N-Hida Processes*

Let us specify the results of Section 1 for the case where E is an interval $T = [a, +\infty[$ of $\mathbb{R}$, a is finite or not, and Y_t has a multiplicity equal to 1. Moreover, in this chapter, Y_t and $\mathscr{F}_t$ will be defined as follows: Let $(\Omega, \mathscr{F}, P, B_t, t \in T)$ be a brownian measure on T, $\mathscr{F}_t = \mathscr{F}_t(B)$ being a complete field generated by B up to t. Then $\{Y_t, t \in T\}$ will be a process such that for every t, Y_t is in $L^2(\Omega, \mathscr{F}_t, P)$ and therefore representable under the form

$$Y_t = \int^t F(t, u)\, B(du), \tag{3.1}$$

$\int^t$ denoting the integral on $T \cap\,]-\infty, t]$ and where

$$F(t, u, \omega) \text{ is } \mathscr{B} \times \mathscr{F} \text{ measurable for every } t \tag{3.2}$$

($\mathscr{B}$ is the borelian field of T), $F(t, u)$ being $\mathscr{F}_u$-measurable,

$$\int^t E(F(t, u)^2)\, du < \infty \qquad \text{for every } t. \tag{3.3}$$

The representation will be said *proper* if, for every t, $\mathscr{F}_t(Y)$, the complete field generated by $\{Y_s, s \leqslant t\}$ is identical to $\mathscr{F}_t = \mathscr{F}_t(B)$. $F(t, u)$ being $\mathscr{F}_u$ measurable, we have

$$E(Y_t \mid \mathscr{F}_s) = \int^s F(t, u)\, B(du) \qquad \text{if} \quad s \leqslant t$$

and therefore, if the representation is proper,

$$E(Y_t \mid \mathscr{F}_s(Y)) = \int^s F(t, u)\, B(du)$$

In such a case the representation is said to be canonical (cf. [5]). In [5], Hida studied the N-ple Markov properties (N-Hida according to our terminology) for processes which can be represented by a deterministic kernel $F(t, u)$. Y_t is then necessarily gaussian, and we find the two following definitions he gives:

DEFINITION 3.1. $\{Y_t, t \in T\}$ will be said to allow a proper representation with a deterministic kernel $(B, F(t, u))$ if Y allows a (3.1) representation with

(a) $F(t, u)$ is a real (deterministic) function, measurable in u and in

$$L^2(t) = L^2(T \cap]-\infty, t]) \qquad \text{for every } t \text{ of } T.$$

(b) for every t of T

$$\mathscr{F}_t(Y) = \mathscr{F}_t(B).$$

DEFINITION 3.2. We shall call an N-order deterministic Goursat kernel a kernel of the form

$$F(t, u) = \sum_{\alpha=1}^{N} f_\alpha(t)\, g_\alpha(u)$$

where

(C1) the deterministic functions f_α form a Tchebychev system on every $[s, +\infty[$ where $s \in T : t_1, t_2, \ldots, t_N$ exist bigger than s such that

$$\det(f_\alpha(t_j))_{\alpha, j=1,N} \neq 0,$$

(C2) the deterministic functions g_α, $\alpha = 1, N$ are measurable and linearly independent in $L^2(t)$ for every $t \in T$.

Hida proves that if a gaussian process allowing a deterministic proper representation is N-Hida, it has a deterministic Goursat kernel representation. We shall use the following generalisation of the Goursat kernel.

DEFINITION 3.3. We shall call an N-order Goursat kernel relative to the family $\{\mathscr{F}_t, t \in T\}$ a kernel of the form:

$$F(t, u) = \sum_{\alpha=1,N} f_\alpha(t)\, g_\alpha(u)$$

where the deterministic f_α satisfy (C1) and the g_α satisfy (C2): $g_\alpha(u, \omega)$ is $\mathscr{B} \times \mathscr{F}$-measurable $g_\alpha(u)$ is $\mathscr{F}_u$-measurable and $g_\alpha 1_{]-\infty,t]}$ are independent in $L^2(P \times du)$ for every $t \in T$.

The following result is inferred from Theorem 1:

THEOREM 3.1. *A necessary and sufficient condition for a process allowing a* (3.1) *representation to be N-Hida relative to $\mathscr{F}_t$ is that its representation be an N-order Goursat kernel relative to $\mathscr{F}_t$. If, furthermore, the kernel is deterministic, the Goursat kernel will be deterministic too. Moreover, the representation is proper iff the process is N-Hida (relative to its own field).*

Proof. Sufficient condition: Let Y_t be represented by an N-order Goursat kernel

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t)\, G_\alpha(t) \qquad \text{where} \quad G_\alpha(t) = \int^t g_\alpha(u)\, B(du). \tag{3.4}$$

Then $G(t) = {}^t(G_1(t), \dots, G_N(t))$ is a regular martingale satisfying (C2) relative to $\mathscr{F}_t$ and therefore (3.4) is a Goursat representation of T_t relative to $\mathscr{F}_t$.

Necessary condition. From Theorem 1 we can infer that Y_t allows an N-order (proper) Goursat representation

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t)\, G_\alpha(t)$$

where $G(t) = {}^t(G_1(t), \dots, G_N(t))$ is a regular $\mathscr{F}_t$-martingale. Since the (f_α) system satisfies (C1), we infer that the $G_\alpha(t)$, $\alpha = 1, N$ are expressed as a linear function of the $E(Y_{t_i} \mid \mathscr{F}_t)$ for some $t_1, t_2, \dots, t_N$ of the future of t. Therefore each $G_\alpha(t)$ is square integrable

$$G_\alpha(t) = \int^t g_\alpha(u)\, B(du),$$

g_α satisfying (C′2) because $G(t)$ is regular and because each $G_\alpha(t)$ belongs to L^2.

If the kernel of Y_t is deterministic then a.s.

$$F(t, u) = \sum_{\alpha=1,N} f_\alpha(t)\, g_\alpha(u). \tag{3.5}$$

On $[u, +\infty[$, N points $t_1, \dots, t_N$ exist such that

$$\det(f_\alpha(t_j)) \neq 0$$

and the inversion of the system obtained by writing the equation above at these N points shows that each $g_\alpha(u)$ can be chosen deterministic. Lastly, if the representation of Y_t is proper, $\mathscr{F}_t = \mathscr{F}_t(Y)$ and therefore the notions of N-Hida and of N-Hida relative to $\mathscr{F}_t$ coincide.

3.2. *Processes with Differentiable Goursat Kernels*

Let $\{Y_t, t \in T\}$ be a square integrable centered process allowing an N-order Goursat kernel representation given by (3.4) and let F be defined by (3.5).

THEOREM 3.2. *Y_t is differentiable in t up to order K iff the f_α are differentiable in t up to order K and if* P–a.s. *for* $k = 0, K-1$,

$$\sum_{\alpha=1,N} f_\alpha^{(k)}(t)\, g_\alpha(t) = 0.$$

We then have

$$Y_t^{(K)} = \sum_{\alpha=1,N} f_\alpha^{(K)}(t)\, G_\alpha(t).$$

Proof. The proof is carried out by recurrence on K. If

$$Y_t^{(K-1)} = \sum_{\alpha-1,N} f_\alpha^{(K-1)}(t)\, G_\alpha(t)$$

we have

$$Y^{(K-1)}(dt) = \left[\sum_{\alpha=1}^{N} f_\alpha^{(K)}(t)\, G_\alpha(t)\right] dt + \left[\sum_{\alpha=1}^{N} f_\alpha^{(K-1)}(t)\, g_\alpha(t)\right] B(dt).$$

An interesting case is met where the $G_\alpha(t)$, $\alpha = 1, N$ can be calculated in function of the Y_t and their derivatives, the simplest case being where Y_t is $N-1$ times differentiable and the $f_\alpha(t)$ satisfy the

(C3) The wronskian matrix Df of f_i, its general term being $(f_\alpha^{(j)}(t))$, is regular for every t.

If we write

$$G_{(t)}^{(N)} = {}^t(G_1(t),\dots, G_N(t))$$
$$Y_{(t)}^{(N)} = {}^t(Y_t,\dots, Y_t^{(N-1)}),$$

the theorem is expressed as

$$Y_{(t)}^{(N)} = (Df)(t)\, G_{(t)}^{(N)}.$$

One concludes that if (Df) is invertible, $Y_{(t)}^{(N)}$ is a base of the space $\mathscr{E}_t$ on which the future is projected.

In [7] Pitt studies the notion of p-markovian processes which in the case of $\mathbb{R}$ is reduced to the following notion:

DEFINITION. The Y_t process will be called p-markovian if the minimal splitting field of $\sigma(Y_s, s < t)$ and of $\sigma(Y_s, s > t)$ is generated by Y_t, $Y_t',\ldots, Y_t^{(p-1)}$.

Let us remind our reader that a splitting of two fields $\mathcal{A}$ and $\mathcal{B}$ is one of the fields $\mathcal{C}$ such that $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent in relation to $\mathcal{C}$.

If Y_t allows a proper representation relative to B and is N-Hida, being moreover $(N-1)$ times differentiable thus satisfying condition (C3) for one of the Goursat kernel differentiable representations, then $\sigma(Y_t,\ldots, Y_t^{(N-1)})$ will be a splitting field of $\sigma(Y_s, s < t)$ and $\sigma(Y_s, s > t)$. Furthermore $Y_t,\ldots, Y_t^{(N-1)}$ are in $\sigma(Y_s, s < t) \cap \sigma(Y_s, s > t)$ and therefore in all splitting fields of these two latter fields. To express the $G_\alpha(t)$, $\alpha = 1, N$ as a function of Y_t and its derivatives all these derivatives up to $N-1$ are needed. Lastly, the N variables $Y_t,\ldots, Y_t^{(N-1)}$ are independent since, for the chosen Goursat representation, the $G_\alpha(t)$, $\alpha = 1, N$ are independent and Df is regular. From there the result is deduced.

THEOREM 3.3. *Let $(Y_t, t \in T)$ be a process allowing a proper representation in B with a deterministic kernel of N-Hida, $N-1$ times differentiable, where the functions f_α, $\alpha = 1, N$ satisfy condition* (C3), $(Df)(t) \neq 0$,

$$Y_t = \int^t \sum f_\alpha(t)\, g_\alpha(u)\, B(du);$$

then

(1) *Y_t is N-markovian for all pairs of open sets $T \cap]t, +\infty[$ and $T \cap]-\infty, t[$,*

(2) *Y_t satisfies the stochastic differential equation*

$$Y^{(N-1)}(dt) - (f_1,\ldots,f_N)^{(N)}\, (Df)(t)^{-1}\, Y(t) = a(t)\, B(dt)$$

where

$$a(t) = \sum_{\alpha=1,N} f_\alpha^{(N-1)}(t)\, g_\alpha(t).$$

We shall assume for the rest of this paragraph and for the following one that

(C4) $a(t) \neq 0$ on T.

Y_t satisfies the stochastic differential equation

$$a^{-1}(t)[Y^{(N-1)}(dt) - (f_1,\ldots,f_N)^{(N)}\, (t)(Df)^{-1}\, (t)\, Y(t)] = B(dt) \qquad (3.5)$$

to which the functional differential equation

$$Ly = h \tag{3.6}$$

is associated, where

$$(Ly)(t) = a^{-1}(t)[y^{(N)}(t) - (f_1,\ldots,f_N)^{(N)}(t)(Df)^{-1}(t)\,{}^t(y_1,\ldots,y^{(N-1)})(t)]. \tag{3.7}$$

If we limit ourselves to the case $T = \mathbb{R}$ (resp. $T = \mathbb{R}^+$) and lay down the initial condition

$$y(t_0) = y'(t_0) = \cdots = y^{(N-1)}(t_0) = 0$$

the equation $(Ly)(t) = h(t)$ has only one solution given by the Riemann kernel:

$$y(t) = \int^t F(t, u)\, h(u)\, du$$

for every ω. The solutions of the deterministic coefficient equation $Ly = 0$ with no second member are $f_1,\ldots,f_N$. The general solution of (3.5) is therefore

$$X_t = Y_t + \sum_{\alpha=1,N} \lambda_\alpha f_\alpha(t)$$

where

$$Y_t = \sum_{\alpha=1,N} f_\alpha(t) \int^t g_\alpha(u)\, B(du)$$

$$\lambda = {}^t(\lambda_1,\ldots,\lambda_N) = (Df)^{-1}(t_1)\,{}^t((X-Y)_{t_1},\ldots,(X-Y)_{t_1}^{(N-1)}).$$

If X satisfies the initial condition where $X_{t_1},\ldots,X_{t_1}^{(N-1)}$ are fixed, we note that X_t is the sum of a purely nondeterministic process Y_t and of a purely deterministic process Z_t (not necessarily orthogonal). Covariance of process Y_t is given by a Dirichlet form defined by operator L when the kernel is deterministic:

Proposition 3.1. *When the Y_t process is associated to a deterministic Goursat kernel, its covariance $r(t, s)$ is given by the Dirichlet form*

$$r(s, t) = \int_T L_u r(t, u)\, L_u r(s, u)\, du.$$

Proof. According to [3], we have

$*$ if $u > t$, $L_u r(t,u) = \int^t F(t,v) L_u F(u,v)\,dv$ but $L_u F(u,v) = \sum_{\alpha=1,N} g_\alpha(v) L_u f_\alpha(u) = 0$ and therefore $L_u r(t,u) = 0$;

$*$ if $u \leqslant t$ then $r(t,u) = \int^u F(t,v) F(u,v)\,dv$ and therefore $L_u r(t,u) = F(t,u)$.

We therefore have for every u, t, $L_u r(t,u) = 1_{\{u<t\}} F(t,u)$ which leads to

$$r(t,s) = \int^{t \wedge s} F(t,v) F(s,v)\,dv = \int_T L_v r(t,v) L_v r(s,v)\,dv.$$

3.3. *Processes Associated to a Stochastic Differential Equation*

Let us consider the following reciprocal problem: let Y_t be a process in $L^2(\mathscr{F}_t)$, $N-1$ times differentiable and the differential equation

$$\frac{1}{a(t)}\left[Y^{(N-1)}(dt) - \left(\sum_{j=1,N-i} a_j(t)\,Y^{(j)}(t)\right)dt\right] = B(dt) \tag{3.8}$$

where $a_j(t)$ are deterministic and continuous, $a(t)$ being $\mathscr{F}_t$-adapted, nonzero, and square integrable. We will confine the case to $T = \mathbb{R}^+$ and suppose moreover that Y satisfies the initial condition

$$Y_0 = Y'_0 = \cdots = Y_0^{(N-1)} = 0. \tag{3.9}$$

If L is the differential operator

$$Ly = a(t)^{-1}\left[y^{(N)} - \sum_{j=1,N-1} a_j(t)\,y^{(j)}\right] \tag{3.10}$$

we have

THEOREM 3.4. *If we assume that equation $Ly = 0$ has N solutions $f_1,\ldots,f_N$, the Df wronskian matrix of which is always regular, then the solution of equation* (3.8) *under the initial conditions* (3.9) *is given by*

$$Y_t = \int_0^t \sum_{\alpha=1}^{N} f_\alpha(t)\,g_\alpha(u)\,B(du)$$

the g_α functions being determinated by the system

$$\sum_{\alpha=1,N} f_\alpha^{(k)}(t)\,g_\alpha(t) = 0, \qquad t = 0, 1,\ldots, N-2$$

$$\sum_{\alpha=1,N} f_\alpha^{(N-1)}(t)\,g_\alpha(t) = a(t).$$

COROLLARY 3.1. *If f_α and g_α respectively verify conditions* (C1) *and* (C2), *then Y_t is N-Hida respectively to $\mathscr{F}_t$.*

Proof. The process $X_t = \sum_{\alpha=1,N} f_\alpha(t) \int^t g_\alpha(u) B(du)$ satisfies Eq. (3.8) and the initial conditions (3.9) if X_t is $N-1$ times differentiable, the term in $B(dt)$ in $X^{(N-1)}(t)$ is $a(t)$, and $(f_1,..., f_N)^{(N)} (Df)^{-1} = (a_{N-1},..., a_0)$; we may write the latter equation as

$$(f_1,..., f_N)^{(N)} = (Df)(a_{N-1},..., a_0)$$

or also $Lf_\alpha = 0$, $\alpha = 1, N$.

Then let f_α, $\alpha = 1, N$ be N solutions with regular wronskian for every t. Then g_α satisfy

$$^t(g_1,..., g_N) = (Df)^{-1}\, {}^t(0,..., 0, a) \tag{3.11}$$

and in this case X_t satisfies (3.8) and therefore

$$t \in \mathbb{R}^+, \qquad \text{P–a.s.} \quad X_t = Y_t.$$

Remarks. (a) If $T = \mathbb{R}$ (or $T = \mathbb{R}^+$) and if the initial conditions of (3.8) are $Y_{t_0},..., Y_{t_0}^{(N-1)}$ known for a $t_0 \in T$, then the solution is $Y_t = X_t + Z_t$, where X_t has the form specified in Theorem 3.4 and Z_t is determinitic, determined by the values of $X - Y$ in t_0. If $T = \mathbb{R}$ and Y_t is purely nondeterministic, the following question may be asked: Do we have $Y_t = X_t$? Or equivalently,

$$\bigcap \mathscr{M}_t(X) = \bigcap \mathscr{M}_t(Y) = 0?$$

(b) Assumption (C3) on f_α, $\alpha = 1$, N implies assumption (C1). But (C3) does not in general imply condition (C′2) on g_α, $\alpha = 1, N$ in the solution to (3.11).

4. PROCESSES ON $\mathbb{R}^2$

The theory developed in Section 1 in the case of a filtering ordered set E is now going to be applied, specified, and developed in the case where $E = \mathbb{R}^2$ with the order introduced in Section 2.

$(\Omega, \mathscr{F}, P)$ will be a probability space with an $\{\mathscr{F}_{st}, (s,t) \in \mathbb{R}^2\}$ filtration to which are associated

$$\begin{aligned} &\{\mathscr{F}_s^1, s \in \mathbb{R}\} \qquad \text{where} \quad \mathscr{F}_s^1 = \mathscr{F}_{st}^1 = \bigvee_{u<s} \mathscr{F}_{uv}, \\ &\{\mathscr{F}_t^2, t \in \mathbb{R}\} \qquad \text{where} \quad \mathscr{F}_t^2 = \mathscr{F}_{st}^2 = \bigvee_{v<t} \mathscr{F}_{uv}, \end{aligned} \tag{4.0}$$

and $\mathscr{G}_{st} = \mathscr{F}_s^1 \vee \mathscr{F}_t^2$.

We shall start by extending the results of Section 3.

4.1. *Representation of N-Hida Prcesses of Multiplicity* 1

W_{st} is a brownian motion on R^2. $\mathscr{F}_{st}$ being the field of W on $R_{st} =]-\infty, s] \times]-\infty, t]$. Let $(Y_{st}, (s,t) \in \mathbb{R}^2)$ be a process of $L^2(P)$ $\mathscr{F}_{st}$-adapted. The representation of Y_{st} will be said proper if, for every (s,t), $\mathscr{F}_{st} = \mathscr{F}_{st}(Y)$ in which case we have $\mathscr{F}^1_s = \mathscr{F}^1_s(Y)$ as well as for the fields $\mathscr{F}^2$ and $\mathscr{G}$.

In the following pages attention will be drawn to Hida properties relative to the $\mathscr{F}_{st}$ filtration, and the Goursat representations will then be relative to the same family, Y_z is in $L^2(\Omega, \mathscr{F}, P)$ for every z of $\mathbb{R}^2$. We therefore have various representations of Y_z as stochastic integrals (cf. Wong and Zakai [11], also [1, 2]).

$\mathscr{F}$-representation

$$Y_z = \int_{R_z} F(z;\xi)\, W(d\xi) + \int_{(R_z)^2} H(z;\xi,\xi')\, W(d\xi)\, W(d\xi') \tag{4.1}$$

where for every z, $F(z;\xi)$ is $\mathscr{F}_\xi$-adapted, while $H(z;\xi,\xi')$ is $\mathscr{F}_{\xi\vee\xi'}$ adapted equal to zero outside $\{\xi \curlywedge \xi'\}$, with, for every z $F(z;\cdot) \in L^2(R_z)$, $H(z;\cdot,\cdot) \in L^2(R_z^2)$.

A representation of this kind is canonical for $\mathscr{F}$ which means that for every $z_0 < z$:

$$E(Y_z \mid \mathscr{F}_{z_0}) = \int_{R_{z_0}} F(z;\xi)\, W(d\xi) + \int_{(R_{z_0})^2} H(z;\xi,\xi')\, W(d\xi)\, W(d\xi')$$

as well as for $\mathscr{F}^1$ and $\mathscr{F}^2$:

$$E(Y_z \mid \mathscr{F}^1_{s_0}) = \int_{R_{s_0,\infty}} F(z;\xi)\, W(d\xi) + \int_{(R_{s_0,\infty})^2} H(z;\xi,\xi')\, W(d\xi)\, W(d\xi').$$

(Remark: the representation will not be canonical relative to the $\mathscr{G}_z$-family, since we then have

$$E(Y_z \mid \mathscr{G}_{z_0}) = \int_{Q_{z_0}} F(z;\xi)\, W(d\xi) + \int_{(Q_{z_0})^2} E(H(z;\xi,\xi') \mid \mathscr{G}_{z_0})\, W(d\xi)\, W(d\xi')$$

where $Q_{z_0} = \{\zeta \in \mathbb{R}^2, \zeta \not> z_0\}$ and $z_0 < z$.)

$\mathscr{F}^1$-representation

$$Y_z = \int_{R_z} F_1(z;\xi)\, W(d\xi) \tag{4.2}$$

where for every z, $F_1(z;\xi)$ is $\mathscr{F}^1_\xi$-adapted in L^2. (We in fact have $F_1(z;\xi) = F(z;\xi) + \int_{R_z} H(z;\xi',\xi)\, W(d\xi')$.)

This representation is canonical relative to $\mathscr{F}_z^1$. If the Y_z process is N-Hida relative to $\mathscr{F}_z$ we may express it $Y_z = \sum_{\alpha=1,N} f_\alpha(z) M_\alpha(z)$, where M_α is a martingale for $\mathscr{F}$. Since $\{f_\alpha\}$ is a Tchebychev system, the M_α martingales are a linear function of $E(Y_{z_i} | \mathscr{F}_z)$ at N points $z_1, \ldots, z_N$ of the future of z. The M_α martingales are in L^2, and according to Wong and Zakai representation theorem ([6]) we have

$$M_\alpha(z) = \int_{R_z} g_\alpha(\xi)\, W(d\xi) + \int_{(R_z)^2} k_\alpha(\xi, \xi')\, W(d\xi)\, W(d\xi')$$

where

$$\text{(C1)} \quad \begin{cases} g_\alpha(\xi) \text{ is in } L^2(\mathscr{F}_\xi),\ k_\alpha(\xi, \xi') \text{ in } L^2(\mathscr{F}_{\xi \vee \xi'}), \\ \text{the latter being equal to zero if we do not have } \xi \curlywedge \xi'. \end{cases}$$

DEFINITION 4.1. Representation (4.1) will be said to be a Goursat bi-kernel representation of Y relative to $\mathscr{F}_z$ if

$$F(z; \xi) = \sum_{\alpha=1,N} f_\alpha(z)\, g_\alpha(\xi)$$

$$H(z; \xi, \xi') = \sum_{\alpha=1,N} f_\alpha(z)\, k_\alpha(\xi, \xi')$$

where (C1) and (C2): the $\{f_\alpha, \alpha = 1, N\}$ system is a Tchebychev system on $\{z, z > z_0\}$ for every z_0.

Lastly, $M(z) = \{M_\alpha, \alpha = 1, N\}$ is a regular martingale.

When g_α, k_α, $\alpha = 1, N$ are deterministic, we shall speak of a deterministic bi-kernel. When all the k_α are equal to zero, we shall speak of a Goursat kernel.

Remark. Let us note that any M_α martingale may be divided into two orthogonal parts: its strong martingale part $(g_\alpha \cdot W)_z$, and $(k_\alpha \cdot WW)_z$. From this we gather that linear independence of M_α, $\alpha = 1, N$ in L^2 will be implied, either by that of g_α, $\alpha = 1, N$ in $L^2(R_z)$ (more precisely, $L^2(\Omega \times R_z, P \times d\xi)$), or by that of K_α in $L^2(R_z^2)$ for every z of R^2.

THEOREM 4.1. (a) *A necessary and sufficient condition for an L^2 process to be N-Hida relative to $\mathscr{F}_z$ is that it be represented by an N order Goursat bi-kernel relative to $\mathscr{F}_z$.*

(b) *If it is expressed by $Y_z = \int_{R_z} F_1(z; \xi)\, W(d\xi)$, where $F_1(z; \xi)$ is $\mathscr{F}_\xi$ adapted, it has an N-order Goursat kernel representation. The M_α martingales are then strong.*

(c) *If, moreover, $F_1(z; \xi)$ is deterministic, it has an N order deterministic Goursat kernel representation: g_α is deterministic, each $M_\alpha(z)$ strong martingale is gaussian.*

Proof. (a) It is derived immediately from what is written above.

(b) We may write, if $z = (s, t)\, F_1(z; \xi) = \sum_{\alpha=1,N} f_\alpha(z)\, L_\alpha(t, \xi)$ where, if $\xi = (u, v)$, $L_\alpha(t, \xi) = g_\alpha(\xi) + \int_{R_{ut}} k_\alpha(\xi', \xi)\, W(d\xi')$.

$g_\alpha(\xi)$ being $\mathscr{F}_\xi$-measurable, one infers that $\sum_{\alpha=1,N} f_\alpha(z) \int_{R_{ut}} k(\xi', \xi)\, W(d\xi')$ is $\mathscr{F}_\xi$-measurable, and therefore equal to its expectation conditional to $\mathscr{F}_\xi$ which is equal to zero: $\int_{R_{ut}} [\sum_{\alpha=1,N} f_\alpha(z)\, k_\alpha(\xi', \xi)]\, W(d\xi') = 0$. Therefore, for every $\xi < z$, $\xi' = (u', v') \not\approx \xi$, $v' \leqslant t$, $\sum_{\alpha=1,N} f_\alpha(z)\, k(\xi, \xi') = 0$.

Let $z_1, \ldots, z_N$ be N points of the future of $\xi \vee \xi'$ at which $\det(f_\alpha \mid z_i) \neq 0$. From the system $\sum_{\alpha=1,N} f_\alpha(z_i)\, k(\xi, \xi') = 0$, $i = 1, N$, we infer $k_\alpha(\xi, \xi') = 0$, $\alpha = 1, N$. Then by writing $F_1(z, \xi) = \Sigma_{\alpha=1,N} f_\alpha(z)\, g_\alpha(\xi)$ for $z = z_1, \ldots, z_N$, we infer that $g_\alpha(\xi)$ is deterministic.

4.2. *Hida Properties Relative to $\mathscr{F}^1$ and $\mathscr{F}^2$ Fields*

DEFINITION 4.2. The process $(Y_z, z \in \mathbb{R}^2)$ will be said N-Hida for the $\mathscr{F}_s^1$ fields, or N-Hida[1], if for every t and every pair (s, s'), $s \leqslant s'$, the space $\operatorname{sp}\{E(Y_{\sigma t} \mid \mathscr{F}_s^1), s' < \sigma\}$ has an N dimension; it will in such a case be denoted $\mathscr{E}_{st}^{(1)}$.

DEFINITION 4.3. Y_z will be said to allow an N-order 1-Goursat representation relative to $\mathscr{F}_s^1$ if it may be expressed as

$$Y_z = \sum_{\alpha=1,N} f_\alpha(z)\, M_\alpha(z)$$

where

(C1) for every t the N functions of s, $f_\alpha(s, t)$, form a Tchebychev system on any $[s_0, +\infty[$;

(C2) $M(z) = {}^t(M_1(z), \ldots, M_N(z))$ is a regular vectorial 1-martingale relative to $\mathscr{F}_s^1$.

Lastly, the 1-representation will be called proper if, $\forall s$, $\mathbb{F}_s^1 = \mathscr{F}_s^1(Y)$.

THEOREM 4.2. *Y_z is N-Hida[1] for $\mathscr{F}_s^1$ iff it has an N order $1-$Goursat representation for $\mathscr{F}_s^1$.*

All we need is to use the proof of Theorem 1, t being set, for the processes in s conditioned by $\mathscr{F}_s^1$.

Case of multiplicity 1. Let us for one instant return to the assumptions made in the former paragraph 4.1. We then show, Y being in L^2, that the $M_\alpha(z)$ 1-martingales are in L^2. They are therefore expressed under the form

$$M_\alpha(z) = \int_{R_z} L(t, \xi)\, W(d\xi) \qquad \text{(cf. [1, 4]).}$$

$L_\alpha(t;\xi)$ being in $L^2(\mathscr{F}_{ut})$ if $\xi=(u,v)$, and if $Y_z=\int_{R_z} F_1(z;\xi)\,W(d\xi)$ (representation (4.2)) we have

$$F_1(z;\xi)=\sum_{\alpha=1,N} f_\alpha(z)\,L_\alpha(t;\xi). \tag{4.3}$$

If $F_1(z;\xi)$ is $\mathscr{F}_\xi$-adapted, so are the N functions $L_\alpha(t;\xi)$. Since N values $u_1,\dots,u_N$ larger than u exist such that $\det(f_\alpha(u_i,t))\neq 0$ the system of equations (4–3) taken in $(z_i=(u_i,t)$ is invertible. If $F_1(z;\xi)$ is deterministic (in which case Y_z is gaussian), the same proof shows that each L_α is deterministic (in which case each M_α 1-martingale is gaussian).

THEOREM 4.3. (a) *A squared integrable centered process, represented in relation to a W brownian measure, is N-Hida*[1] *for* $\mathscr{F}^1_s$ *iff it allows a 1-Goursat representation relative to* $\mathscr{F}^1_s$ *where*

$$M_\alpha(z)=\int_{R_z} L_\alpha(t,\xi)\,W(d\xi),$$

$L_\alpha(t;\xi)$ *being* $\mathscr{F}^1_\xi$*-adapted.*

(b) *If* $Y_z=\int_{R_z} F_1(z;\xi)\,W(d\xi)$ *and if* $F_1(z;\xi)$ *is* $\mathscr{F}_\xi$*-adapted, the same will apply to each* L_α.

Hida[1] *and Hida*[2] *processes.* Once again $(\Omega,\mathscr{F},\mathscr{F}_z,P)$ is general. It would be a mistake to think that a process satisfying the N-Hida property relative to the $\mathscr{F}^1_z$ and $\mathscr{F}^2_z$ fields always satisfies this property relative to the $\mathscr{F}_z$ fields.

COUNTER EXAMPLE. Consider the kernel defined on $]-1,+\infty[^2$:

$$\begin{aligned} F(s,t;\ u,v) &= 2(t+u)+2(s-t)(1+v)\\ &= (1+s-t)(s+u+v)+(1-s+t)(s+u-v). \end{aligned}$$

One can prove that the associated process is 2-Hida[1], 2-Hida[2], but 3-Hida. In the same way if Y_z is a process at the same time that is both p-Hida[1] and q-Hida[2], then it is in general not true that Y_z is N-Hida or even Hida-bounded for the $\mathscr{F}_z$ fields (cf. [10] for a counter example).

Throughout the end of this paragraph and in the two following ones, the fields $\mathscr{F}^1_s$ and $\mathscr{F}^2_t$ are conditionally orthogonal relative to $\mathscr{F}_{st}$ if we write

$$E^1_s=E(\cdot\mid\mathscr{F}^1_s)\qquad E^2_t=E(\cdot\mid\mathscr{F}^2_t)\qquad E_{st}=E(\cdot\mid\mathscr{F}_{st}).$$

Assumption (H): $E_{st}=E^1_s\circ E^2_t=E^2_t\circ E^1_s$.

THEOREM 4.4. *If (p, q) is a pair of integers and if Y_z satisfies the assumptions*

* *Y_z is Hida[1]-bounded by p, that is to say such that $\forall (s, t)$, $\dim \mathscr{E}^{(1)}_{st} \leqslant p$,*

* *for every t_0, t'_0, $t_0 \leqslant t'_0$, there exist $t_1, \dots, t_q$ larger than t'_0 such that $\forall s$, $\{E(Y_{st_j} \mid \mathscr{F}^2_{t_0}), j = 1, q\}$ generates $\mathscr{E}^{(2)}_{st_0}$,*

then Y_z is Hida, bounded by $p \cdot q$, relative to $\mathscr{F}_z$.

This theorem will be useful in Part II [9] and is the key to the study of $\mathscr{G}$-markovian processes.

Proof. Let us have

$$\mathscr{E}^{(1)}_{s_0 t} = \mathrm{sp}\{E(Y_{s't} \mid \mathscr{F}^1_{s_0}), s_0 \leqslant s'\} = \mathrm{sp}\{E(Y_{s't} \mid \mathscr{F}_{s_0}), s_0 \leqslant s'\};$$

then $\dim \mathscr{E}^{(1)}_{s_0 t} \leqslant p$. Whatever $t_0 \leqslant t'_0 \leqslant t_1 \leqslant \cdots \leqslant t_q$ may be chosen such that $\forall u$, $\{E(Y_{ut_j} \mid \mathscr{F}^2_{t_0}), j = 1, q\}$ generates $\mathscr{E}^{(2)}_{s_0 t_0}$. Let $\xi = (u, v)$ be a point of the future of $z_0 = (s_0, t_0)$. We have

$$E(Y_\xi \mid \mathscr{F}_{z_0}) = E[E(Y_\xi \mid \mathscr{F}^2_{t_0}) \mid \mathscr{F}^1_{s_0}].$$

$E(Y_\xi \mid \mathscr{F}^2_{t_0})$ being in $\mathscr{E}^{(2)}_{ut}$, we can write

$$E(Y_\xi \mid \mathscr{F}^2_{t_0}) = \sum_{j=1,q} \lambda_j E(Y_{ut_j} \mid \mathscr{F}^2_{t_0})$$

and therefore

$$\begin{aligned} E(Y_\xi \mid \mathscr{F}_{z_0}) &= \sum_{j=1,q} \lambda_j E[E(Y_{ut_j} \mid \mathscr{F}^2_{t_0}) \mid \mathscr{F}^1_{s_0}] \\ &= \sum_{j=1,q} \lambda_j E[E(Y_{ut_j} \mid \mathscr{F}^1_{s_0}) \mid \mathscr{F}^2_{t_0}]. \end{aligned}$$

Y_z being Hida[1]-bounded by p, whatever $s'_0 \geqslant s_0$, at most p values $s_{1j}, \dots, s_{p_j j}$ (where $p_j \leqslant p$) may be chosen for each t_j such that $E(Y_{s_{kj} t_j} \mid \mathscr{F}^1_{s_0})$, $k = 1, p_j$ generate $\mathscr{E}^{(1)}_{s_0 t_j}$. $E(Y_{ut_j} \mid \mathscr{F}^1_{s_0})$ being in this space, it is a linear combination of this generator. Lastly,

$$E(Y_\xi \mid \mathscr{F}_{z_0}) = \sum_{j=1,q} \lambda_j \sum_{k=1,p_j} \mu_{jk} E(Y_{s_{kj} t_j} \mid \mathscr{F}_{s_0 t_0}).$$

We have therefore exhibited a generator with at the very most $p.q.$ vectors of the $\mathscr{E}_{z_0}$ space. $\dim \mathscr{E}_{z_0} \leqslant p.q.$ Moreover, the choice of s'_0 and t'_0 being arbitrary, we were able to choose this generator in the future of any point z'_0 of the future of z_0. Therefore Y_z is Hida-bounded by $p.q.$

Remarks. (a) If furthermore we suppose that Y_z is p-Hida[1] and q-Hida[2] then

$$\dim \mathscr{E}_{z_0}^{(1)} = p \leqslant \dim \mathscr{E}_{z_0}$$
$$\dim \mathscr{E}_{z_0}^{(2)} = q \leqslant \dim \mathscr{E}_{z_0}$$

and therefore $\max\{p, q\} \leqslant \dim \mathscr{E}_{z_0} \leqslant p \cdot q$. In particular, if $p = 1$, Y_z will be q-Hida.

(b) Let us denote $d_z = \dim \mathscr{E}_z$. We know that d_z increases and has $p \cdot q$ upper bound. Denote $N = \sup d_z$ and $S = \{z \in \mathbb{R}^2, d_z = N\}$. Then S is nonempty and satisfies $(z \in S, z < z') \Rightarrow z' \in S$, and on S, Y is N-Hida.

(c) If Y_z is separately stationary in s and in t, we have

$$\forall (h, k) \in \mathbb{R}^2, \qquad \dim \mathscr{E}_{st} = \dim \mathscr{E}_{s+h,t} = \dim \mathscr{E}_{s,t+k}$$

and therefore Y_z is N-Hida.

If Y_z is stationary in the direction (h, k), $h > 0$, $k > 0$, take z_0 such that $d_{z_0} = N$. Whatever z, a real number l exists such that $z_0 < z + l(h, k)$. Therefore $d_z = d_{z+l(h,k)} = N$ and Y_z is N-Hilda.

Examples of p-Hida[1] and q-Hida[2] processes can be given which are not constantly Hida on all $\mathbb{R}^2$ (cf. [10]).

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